NONUNIQUENESS OF THE STATIONARY SOLUTION TO COMBUSTION-THEORY EQUATIONS WITH A CONSTANT RATIO OF THERMAL CONDUCTIVITY AND DIFFUSION COEFFICIENTS

R. D. Bachelis and V. G. Melamed

Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 1, pp. 161-166, 1968

The authors consider the stationary solution to a one-dimensional process of gas-mixture combustion; the usual assumptions dealing with thermal-conductivity and thermal diffusivity coefficients and reaction rate are used.

Despite current opinion that the stationary solution is unique (this is supported by a uniqueness proof of the some particular cases), the uniqueness problem is still unsolved in the general case. In this article, by constructing a contradictory example, we show that uniqueness does not exist when usual restrictions are followed. As a result of this study, it is shown that it is possible to choose the reaction rate and thermal-conductivity and diffusion coefficients as functions of temperature so that the combustion-theory equations have at least two solutions satisfying all the stated conditions.

§1. The stationary solution to gas-mixture combustion in one dimension [1] is described by

$$\lambda \frac{du}{dy} = \frac{d}{dy}\left[\alpha_1(u)\frac{du}{dy}\right] + F(u)c, \qquad (1.1)$$
$$\lambda \frac{dc}{dy} = \frac{d}{dy}\left[\alpha_2(u)\frac{dc}{dy}\right] - F(u)c,$$

for the conditions

$$u(-\infty) < u(y) < u(\infty) = 0, \qquad u(-\infty) = u_-$$
$$c(-\infty) > c(y) > c(\infty) = 0, \qquad c(-\infty) = c_0 > 0.$$

The number $u_- < 0$ has previously been specified. Here u is the mixture temperature, c is the concentration of active material, F(u)c is the reaction rate, $F(u) \equiv 0$ for $u \in [u_-, u_0]$, $F(u) > 0$ for $u > u_0$, $\alpha_1(u) > 0$ is the thermal-conductivity coefficient, $\alpha_2(u) > 0$ is the diffusion coefficient, and $\lambda = \text{const} > 0$.

The first term in (1.1) has an integral from which it follows that

$$c_0 = -u_-.$$

It is easy to show that $u'(y) > 0$; therefore, by taking u as the independent variable, (1.1) can be represented as

$$v' = \lambda - \frac{f(u)c}{v}, \qquad c' = \beta\left[\frac{\lambda}{v}(c+u) - 1\right],$$

for conditions

$$v(u_-) = 0, \qquad v(0) = c(0) = 0.$$

Here

$$v(u) = \alpha_1 u' > 0, \qquad \alpha_1 F(u) = f(u), \qquad \alpha_1 / \alpha_2 = \beta(u) > 0,$$

Extensive study has been devoted to the existence and uniqueness of system (1.2)-(1.3). In the general case, the uniqueness of the solution is still an unsolved problem. Recently, the authors proved that it is possible to choose functions $f(u)$ and $\beta(u)$ such that (1.2) has a nonunique solution [2]. In [3], it was shown that when $\beta \equiv$ const, solutions to (1.2) exist for all $\beta > 0$ and unique for all $\beta > 1$. It is clear that the methods used in [3] can also prove uniqueness for the variables $\beta(u)$ given the condition that $\beta(u) \geq 1$ for all u. Thus, uniqueness of the solution for $\beta \equiv \text{const} < 1$ is still an open problem.

In this article, it is shown that for all $\beta = \text{const} \in (0.1)$ we can choose an $f(u)$ satisfying the above conditions so that (1.2) has at least two solutions. It is also shown that $f(u)$ can be made continuously differentiable any previously specified number of times.*

Evidently, a similar choice of $f(u)$ is also possible for any variable $\beta(u)$ whose values are less than unity for some u. Thus, for $\beta(u) \equiv \text{const} < 1$ or variables $\beta(u)$ not satisfying inequality $\beta(u) \geq 1$ for some u, uniqueness is defined by function $f(u)$ and requires special study for each particular case. Physical realizability of function $f(u)$ is not considered in this study.

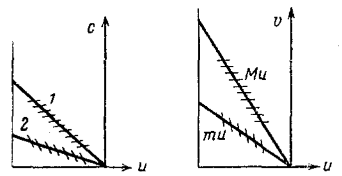

Fig. 1

Lemma 1.1. Assume the following differential equation is specified.

$$y' = \varphi(x, y, \ldots). \qquad (1.4)$$

The remaining functions in the system are described by the dots (Fig. 1 if the given equation is not independent. Let some line $y = y(x)$ be specified.

We introduce the following notation: ordinary derivative $Y'(x)$ represents the natural slope $Y^{(0)'}$ and the term $\varphi(x, Y, \ldots)$ represents the slope of the integral lines $Y^{1'}$. It is clear that if there are no singular points on some interval (a, b) and $Y^{(0)'} > Y^{1'}$, for motion from right to left, no integral curve in (1.4) can intersect the line $Y = Y(x)$ from below. Similar facts can easily be established for motion along the integral line from left to right and for an inequality with the opposite sign.

Consider (1.2) and the line $c = -u$ for $u < 0$. Then, for $\beta \in (0.1)$, we have $c^{0'} = -1$, and $c^{1'} = -\beta > c^{(0)'}$.

As will be shown below, $c'(0) > -1$. Since $c(0) = 0$, there exists a left semineighborhood of point $u = 0$ in which $c(u) < -u$. Therefore, for motion from right to left from any point in this semineighborhood, the indicated inequality holds until $v(u) > 0$. According to (1.2),

$$c'(u) < 0, \qquad u < 0. \qquad (1.5)$$

We now have $c(u) > 0$ for $u < 0$. By allowing for the obtained inequality it is easy to show that when $u < u_0$ we have $v(u) > 0$ for all u for which $f(u) > 0$.

§2. We now construct an example of nonuniqueness when function $f(u)$ is continuously differentiable any previously specified number of times; here $f(u) \equiv 0$ for $u \in [u_{-1}, u_0]$, and $f(u) > 0$ for $u > u_0$. Let any $\beta \in (0,1)$ be specified. We then have:

Lemma 2.1. Let m and M be numbers satisfying inequality $M < m < 0$ and

$$f^{(0)} = (\lambda - m)mQ, \qquad f^1 = (\lambda - M)Mq, \qquad (2.1)$$
$$Q = (M - \lambda\beta) / (\lambda - M)\beta,$$
$$q = (m - \lambda\beta) / (\lambda - m)\beta.$$

Then:

$$0 < f^{(0)} < f^1. \qquad (2.2)$$

(1) If

$$f^{(0)} < f(u) < f^1 \qquad (2.3)$$

on [1, 0] then† on [−1, 0] we have

$$Mu < v(u) < mu, \quad q^{-1}u < c(u) < Q^{-1}u. \qquad (2.4)$$

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*Construction of an example of nonuniqueness can be made much simpler if $f(u)$ is piecewise-constant.

† We can take any negative number as the left boundary of interval [−1, 0].

Proof of (1). From (2.1) it follows that

$$0 < (\lambda - m) / (\lambda - M) < 1, \ 0 <$$
$$< m (M - \lambda\beta) < M (m - \lambda\beta)$$

which in turn yields (2.2).

(2). The point u = 0 is a singular point for (1.2), in view of (1.3). Applying L'Hôpital's rule we obtain a system of equations for finding v'(0) and c'(0). This last equation leads to the Coulomb equation in v'(0). Two of its roots (positive) are not consistent with the physical sense of the problem [3]. The remaining roots are given by

$$v'(0) = {}^1\!/_2 [\lambda\beta - \sqrt{\lambda^2\beta^2 + 4f(0)\beta}] < 0,$$
$$c'(0) = \beta[\lambda - v'(0)] \, [v'(0) - \lambda\beta]^{-1}.$$

Using (2.1) and the upper bound (2.3) we have

$$v'(0) > \lambda\beta - \sqrt{\lambda^2\beta^2 + 4(\lambda - M) M q}.$$

Using (2.1) and lower bound (2.3) we have $v'(0) < m$. It is also easy to verify that

$$Q^{-1} < c'(0) < q^{-1}.$$

From the inequalities obtained for c'(0) and v'(0) it follows that there exists a left semineighborhood of point u = 0, $-\Delta < u < 0$, in which (2.4) holds. We shall prove that (2) of this lemma is satisfied. If $\Delta \geq 1$, it is clear that (2) is true. Let $\Delta < 1$.

Consider the region (Fig. 1)

$$D\{-1 \leqslant u \leqslant \Delta, \ Mu \leqslant v \leqslant mu, \ q^{-1}u \leqslant c \leqslant Q^{-1}u\}.$$

It is easy to verify that if (2.2) holds, we have the following expression for point $(u, v, c) \in D$:

$$v'(u) < m \ \text{ for } \ v = mu, \quad v'(u) > M \ \text{ for } \ v = Mu,$$
$$c'(u) > Q^{-1} \ \text{ for } \ c = Q^{-1}u, \quad c'(u) < q^{-1} \ \text{ for } \ c = q^{-1}u.$$

Note that c is contained in some interval; therefore, the slipes v'(u) and c'(u) (shown in Fig. 1) of the integral lines for vz' v = mu, v = Mu, $c = q^{-1}u$, and $c = Q^{-1}u$, corresponding to the above inequalities, should not be treated as being uniquely defined. Now, on the basis of Lemma 1.1, it follows that (2.4) holds on [−1, 0]. Lemma 2.1 is therefore proven.

Let λ_1, λ_2, and m_1 be any numbers satisfying the inequality

$$\lambda_2 > \lambda_1 > 0, \qquad m_1 < 0.$$

Let m_2 be the negative root of

$$m_1 (m_1 - \lambda_1\beta) = m_2 (m_2 - \lambda_2\beta). \tag{2.5}$$

Its existence and uniqueness are clear. Converting (2.5) we obtain

$$m_1 - m_2 = -m_1\beta (\lambda_2 - \lambda_1) / (m_1 + m_2 - \lambda_2\beta).$$

It now follows that $m_2 < m_1$.

We choose any $M_2 \in (m_1, m_2)$. Finally, let M_1 be a negative root of the equation

$$M_1 (M_1 - \lambda_1\beta) = M_2 (M_2 - \lambda_2\beta). \tag{2.6}$$

Since $M_2 < m_2 < 0$,

$$M_2 (M_2 - \lambda_2\beta) > m_2 (m_2 - \lambda_2\beta).$$

Now, in turn,

$$M_1 (M_1 - \lambda_1\beta) > m_1 (m_1 - \lambda_1\beta), \quad \text{or} \quad M_1 < m_1.$$

Thus, we have

$$M_1 < m_1 < M_2 < m_2 < 0 \tag{2.7}$$

Consider the intervals $f_i^{(0)}$, and f_i^1 $(i = 1,2)$, where

$$f_i^{(0)} = (\lambda_i - m_i) m_i Q_i, \quad f_i^1 = (\lambda_i - M_i) M_i q_i.$$

The fact that each of these intervals is not empty follows from (1) of Lemma 2.1. We shall show that their intersection is also not empty. To do this, we consider

$$f_i^{(0)} / f_i^1 = m_i M_i.$$

From (2.5) and (2.6) it follows that $f_1^{(0)} f_1^1 = f_2^{(0)} f_2^1 = d^2 > 0$. We now have

$$d \in (f_1^{(0)}, f_1^1), \ d \in (f_2^{(0)}, f_2^1).$$

Therefore, the intersection of $(f_1^{(0)}, f_1^1)$ and $(f_2^{(0)}, f_2^1)$ is not empty. We let $(f^{(0)*}, f^{1*})$ be the intersection of the indicated intervals. Let $f(u)$ by any function such that for $u \in [-1, 0]$,

$$f(u) \in (f^{(0)*}, f^{1*}). \tag{2.8}$$

Applying Lemma 2.1 for $\lambda = \lambda_i$ (i = 1, 2) we obtain inequalities for $v_i(u)$ and $c_i(u)$ on (−1, 0). We shall give those inequalities which will be used below:

$$v_2(-1) < -M_2, \ -M_1 > v_1(-1) > -m_1,$$
$$c_2(-1) > -q_2^{-1} > 0, \ c_1(-1) < -Q_1^{-1}. \tag{2.9}$$

§3. In this section we shall consider the solution to (1.2)–(1.3) for $\lambda = \lambda_2$ on $(u_2, -1)$ where $u_2 = -1 + M_2/\lambda_2$, assuming (2.8) is satisfied. From (1.5) it follows that

$$c_2(u) < 1 - M_2 / \lambda_2. \tag{3.1}$$

We choose any $\varepsilon > 0$ and $n > 1$. The importance of n will be shown at the end of the section. We construct

$$v_2^* = \varepsilon + \lambda_2 \left(u + 1 - \frac{M_2}{\lambda_2}\right)\left(1 + \frac{\varepsilon}{2M_2}\right) -$$
$$- \frac{\varepsilon}{2}\left(\frac{u + 1 - M_2/\lambda_2}{-M_2/\lambda_2}\right)^n.$$

We can verify that the function $v_2^*(u)$ possesses the following properties:

$$v_2^*(u) > 0, \ v_2^{*\prime}(u) < \lambda_2, \ v_2^*(u) \leqslant \lambda_2 (u - u_2) + \varepsilon,$$
$$v_2^*(-1) = -M_2 \tag{3.2}$$

Let, for $u \in [u_2, -1]$,

$$f(u) \in (0, \varphi(u)), \ \varphi(u) = (\lambda_2 - v_2^{*\prime}) v_2^* / (1 - M_2/\lambda)_2. \tag{3.3}$$

From the obtained properties for $v_2^*(u)$ it follows that the interval for $f(u)$ is not empty for $u \in [u_2, -1)$. We shall show that for $u \in [u_2, -1)$,

$$v_2(u) < v_2^*(u). \tag{3.4}$$

In fact, according to (2.9) and (3.2),

$$v_2(-1) < -M_2 = v_2^*(-1).$$

Moreover, consider $v_2^*(u)$ in the field of integral lines (1.2). We have

$$v_2^{*1\prime} = \lambda_2 - f(u) \, c_2 / v_2^* > v_2^{*\prime} = v_2^{*0\prime}$$

according to (3.2) and (3.3). Applying lemma 1.1, we obtain (3.4). Using (3.2), we obtain

$$v_2(u) < \varepsilon + \lambda_2 (u - u_2). \tag{3.5}$$

consider the second equation in (1.2) for $\lambda = \lambda_2$ and allow for (3.5):

$$c_2'(u) \leqslant \beta \left\{\frac{\lambda_2 [c_2(u) + u]}{\varepsilon + \lambda_2 (u - u_2)} - 1\right\}.$$

Upon converting this equation to

$$[\ln (c_2 + u_2 - \varepsilon / \lambda_2)]' \geqslant [\ln (u + \varepsilon / \lambda_2 - u_2)^\beta]'$$

and integrating from u_2 to −1, we obtain

$$c_2(u_2) > \frac{\varepsilon - \lambda_2 u_2}{\lambda_2}\left\{1 - \left[\frac{\varepsilon}{\varepsilon - \lambda_2(1 + u_2)}\right]^\beta\right\}.$$

Here, $c(-1) > 0$. Allowing for (1.5) we obtain for all $u \leq u_2$ the solution

$$c_2(u) \geqslant \frac{1}{\lambda_2}\left\{\varepsilon + \lambda_2 - M_2 + (\lambda_2 u_2 - \varepsilon)\left[\frac{\varepsilon}{\varepsilon - \lambda_2(1 + u_2)}\right]^\beta\right\} =$$
$$= c_2^{(0)} > 0,$$

independent of the still unspecified value of $f(u)$.

Remark. For $u = -1$, it follows from (3.3) that

$$f(-1) < \frac{\lambda_2^2 \varepsilon (n+1)}{2(\lambda_2 - M_2)}.$$

Clearly, we can choose an $n > 1$ such that

$$\frac{\lambda_2^2 \varepsilon (n+1)}{2(\lambda_2 - M_2)} > f^{(0)*}$$

The last fact makes it possible to "join" together (2.8) and (3.3) at point $u = -1$.

§4. Now consider the solution to (1.2)–(1.3) for $\lambda = \lambda_1$ on $[u_0, -1)$ where $u_0 = -1 + m_1/\lambda_2$.

In view of (2.7), we have $u_0 < u_2$. No matter what the value of $f(u)$ on (u_0, u_2) (except for the fact that it must not be negative), $v_1^1 \leq \leq \lambda_1$.

According to (2.9), we have the following expression for $u \leq -1$:

$$v_1(u) > \lambda_1 (u+1) - m_1 \tag{4.1}$$

and, in turn,

$$c_2(u_0) < 1 - \frac{m_1}{\lambda_1} + \left[\frac{m_1}{\lambda_2} - 1 - Q_1^{-1}\right]\left(1 - \frac{\lambda_1}{\lambda_2}\right)^{\beta} <$$
$$< \left(1 - \frac{m_1}{\lambda_1}\right)\left[1 + \frac{m_1}{\lambda_1\beta - m_1}\left(1 - \frac{\lambda_1}{\lambda_2}\right)^{\beta}\right] = C_1^{(0)} > 0.$$

In view of (1.5) we obtain $0 < c_1(u) < C_1^{(0)}$ for all $u \in (u_0, 0)$.

We shall show that it is possible to choose numbers λ_1, λ_2, m_1, M_2, and ε without violating previously constructed requirements so that

$$\lambda_2^{\,2} C_1^{(0)} < \lambda_1^2 C_2^{(0)}. \tag{4.2}$$

We choose any m_1 and $\lambda_1 > 0$. Consider the auxiliary expression

$$\lim_{\lambda_2 \to \lambda_1 + 0} \frac{(\lambda_2 - \lambda_1)^{\beta}\,[\lambda_2^2\lambda_1 + \lambda_1^2\lambda_2 - m_1(\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1^2)]\,(\lambda_1\beta - m_1)}{-m_1(\lambda_1 - m_1)\,\lambda_2^{3-\beta}} = 0.$$

Therefore, there exists as$\lambda_2 > \lambda_1$ such that

$$1 > \frac{(\lambda_2 - \lambda_1)^{\beta}\,[\lambda_2^2\lambda_1 + \lambda_1^2\lambda_2 - m_1(\lambda_2^2 + \lambda_1\lambda_2 + \lambda_1^2)\,(\lambda_1\beta - m_1)}{-m_1(\lambda_1 - m_1)\,\lambda_2^{3-\beta}},$$

or, after some transformations,

$$\frac{\lambda_1^2}{\lambda_2^2} > \frac{(1 - m_1/\lambda_1)\,[1 + m_1(1 - \lambda_1/\lambda_2)^{\beta}(\lambda_1\beta - m_1)^{-1}}{1 - m_1/\lambda_2}.$$

Moreover, consider

$$\lim_{(\varepsilon \to 0+0),\ M_2 \to m_1 + 0} \frac{[1 + m_1(1 - \lambda_2/\lambda_1)^{\beta}(\lambda_1\beta - m_1)^{-1}]\,(1 - m_1/\lambda_1)\,\lambda_2}{\varepsilon + \lambda_2 - M_2 + (\lambda_2 u_2 - \varepsilon)\,\varepsilon^{\beta}\,[\varepsilon - \lambda_2\ (1 + u_2)^{\beta}} =$$
$$= \left(1 - \frac{m_1}{\lambda_1}\right)\left[1 + m_1\left(1 - \frac{\lambda_1}{\lambda_2}\right)(\lambda_1\beta - m_1)^{-1}\right]\left(1 - \frac{m_1}{\lambda_2}\right) < \left(\frac{\lambda_1}{\lambda_2}\right)^2$$

Therefore, there exists an $\varepsilon > 0$ and $M_2 \in (m_1, m_2)$ such that for specified m_1, λ_1, and λ_2 we have (4.1).

§5. We now find the upper estimate for $v_1(u_2)$. According to (4.1),

$$v_1(u) > \lambda_1 M_2/\lambda_2 - m_1.$$

Now, from (1.2),

$$v_1' > \lambda_1 - \frac{\lambda_2 \sup\{\varphi(u), C_1^{(0)}\}}{\lambda_1 M_2 - m_1\lambda_2} = h, \quad u \in [u_2, -1),$$

and, from (2.9),

$$v_1(u_2) < H \quad (H = hM_2\lambda_2^{-1} - M_1). \tag{5.1}$$

§6. Thus, as a result of the above construction, it is clear that on $[u_0, u_2]$,

$$0 < c_1(u) < c_1^{(0)},\ c_2(u) > C_2^{(0)} > 0,\ C_2^{(0)}\lambda_1^2 > C_1^{(0)}\lambda_2^2,$$

where no restrictions on imposed on the choice of $f(0)$ for $u \in [u_0, u_2)$ except that it is not be negative.

In this section we shall show that for some added restrictions on $f(u)$ and for $u \in [u_0, u_2)$,

$$v_2(u_0)/\ v_1(u_0) > \lambda_2/\lambda_1. \tag{6.1}$$

We estimate $v_2(u)$ from below. Let

$$\delta \in (0, {}^1/_2\,(u_2 - u_0)\ ,\quad k_2 > 0,$$

and let, for $u \in [u_0 + \delta, u_2 - \delta]$,

$$f(u) > \lambda_2 \sqrt{2k_2(u_0 - u_2 - 2\delta)}\,C_2^{(0)^{-2}} + k_2 = l. \tag{6.2}$$

For the same u, consider

$$V_2(u) = \sqrt{-2k_2C_2^{(0)}(u - u_2 + \delta)}$$

in the field of integral lines (1.2) for $\lambda = \lambda_2$. Here

$$V_2^{(0)'} = -k_2C_2^{(0)}V_2^{-1},\ V_2^{1'} = \lambda_2 - f(u)C_2(u)V_2^{-1}.$$

Then

$$V_2^{0'} - V_2^{1'} = -\lambda_2 + V_2^{-1}(u)\,[f(u)\,C_2(u) - k_2C_2^{(0)}] >$$
$$> -\lambda_2\,[-1 + \sqrt{-2k_2C_2^{(0)}(u - u_2 + \delta)}]\,V_2^{-1}(u) = 0$$

Since $V_2(u_2 - \delta) = 0$ and $v_2(u_2 - \delta) > 0$, according to lemma 1.1 for $u \in [u_0 + \delta,\ u_2 - \delta]$, we have

$$v_2(u) > V_2(u)$$

In particular,

$$v_2(u_0 + \delta) > \sqrt{-2k_2C_2^{(0)}(u_0 - u_2 + 2\delta)} \quad \text{for } u = u_0 + \delta.$$

Moreover, for $u \in [u_0, u_0 + \delta)$, we have

$$v_2'(u) \leqslant \lambda_2,$$

since $f(u)$ is not negative. We now obtain

$$v_2(u_0) > \sqrt{-2k_2C_2^{(0)}(u_0 - u_2 + 2\delta)} - \lambda_2\delta. \tag{6.3}$$

Let some $k_1 > 0$ be specified and let

$$f(u) < k_1 \tag{6.4}$$

for $u \in [u_0, u_2)$.

The estimate from below which must be applied to k_1 in order that the last inequality not contradict (6.2) will be discussed later.

We now estimate $v_1(u)$ from above. To do this we consider the auxiliary function

$$V_1(u) = \sqrt{H^2 - 2k_1C_2^{(0)}(u - u_2)}$$

in the field of integral lines (1.2) for $\lambda = \lambda_1$. We now have

$$V_1^{0'} = -k_1C_1^0V_1^{-1} < \lambda_1 - f(u)C_1(u)V_1^{-1} = V_1^{1'}.$$

According to (5.1),

$$v_1(u_2) < H = V_1(u_2).$$

Now, according to lemma 1.1, we have $v_1(u) < V_1(u)$ for $u \in [u_0, u_2]$. In particular,

$$v_1(u_0) < V_1(u_0) = \sqrt{H^2 - 2k_1C_1^{(0)}(u_0 - u_2)}. \tag{6.5}$$

We shall prove the existence of k_1 and k_2 such that

$$k_1 > k_2 + \lambda_2\sqrt{-2k_2(u_0 - u_2 - 2\delta)}/\sqrt{C_2^{(0)}},$$
$$\frac{\sqrt{-2k_2C_2^{(0)}(u_0 - u_2 + 2\delta)} - \lambda_2\delta}{\sqrt{H^2 - 2k_1C_1^{(0)}(u_0 - u_2)}} > \frac{\lambda_2}{\lambda_1}. \tag{6.6}$$

The first inequality ensures that an $f(u)$ satisfying both (6.1) and (6.3) exists. We choose a sufficient small δ and a θ_1 such that

$$\theta = C_2^{(0)}\lambda_1^2(u_2 - u_0 + 2\delta)\,[C_1^{(0)}\lambda_1^2(u_2 - u_0)]^{-1} > 1,$$

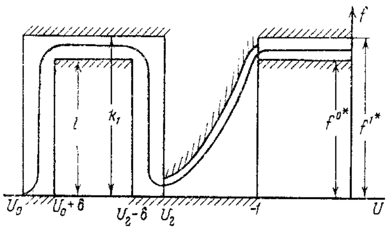

Fig. 2

$$\theta_1 \in (1, \theta).$$

We set $k_1 = \theta k_2$ and show that for sufficiently large k_2, Eq. (6.6) will be satisfied. Consider

$$\lim_{k_2\to\infty} \frac{\sqrt{-2k_2C_2^{(0)}(u_0-u_2+2\delta)}-\lambda_2\delta}{\sqrt{H^2-2\theta_1k_2C_1^{(0)}(u_0-u_2)}} = \frac{\lambda_2}{\lambda_1}\left(\frac{\theta}{\theta_1}\right)^{1/2} > \frac{\lambda_2}{\lambda_1},$$

$$\lim_{k_2\to\infty} \frac{k_2+\lambda_2\sqrt{-2k_2(u_0-u_2+2\delta)}\,(C_2^{(0)})^{-1/2}}{\theta_1k_2} = \frac{1}{\theta_1} < 1.$$

Thus, for sufficiently large k_2, inequality (6.6) is satisfied. Equation (6.1) now follows (6.3), (6.5), and (6.6).

§7. Figure 2 shows all inequalities satisfied by $f(u)$. Clearly, we can construct a function $f(u)$ satisfying all of the indicated inequalities, which is continuously differentiable any specified number of times, and which becomes zero for $u = u_0$ upon contact with any previously specified order. As was shown, the solutions $v_1(u)$, $c_1(u)$ and $v_2(u)$ to (1.2)–(1.3) corresponding to λ_1 and λ_2 satisfy the inequality

$$0 < v_1(u_0)/\lambda_1 < v_2(u_0)/\lambda_2.$$

We choose any

$$q \in (v_1(u_0)/\lambda_1,\ v_2(u_0)/\lambda_2).$$

We set $u_- = u_0 - q$. Then

$$\lambda_1(u_0 - u_-) > v_1(u_0),$$
$$\lambda_2(u_0 - u_-) < v_2(u_0).$$

§8. Using a method similar to that used in [3] for proving existence of a solution it is easy to establish that $\lambda_0 \in (0, \lambda_1)$ and $\lambda_3 \in (\lambda_2, \infty)$ exist such that

$$\lambda_0(u_0 - u_-) = v_0(u_0), \quad \lambda_3(u_0 - u_-) = v_3(u_0). \qquad (8.1)$$

Here $v_0(u)$ and $v_3(u)$ are solutions to (1.2)–(1.3) for, respectively, λ_0 and λ_3.

For $u \in [u_-, u_0]$, we have $f(0) \equiv 0$; therefore, $v_i' = \lambda_i$ and, consequently, it follows from (8.1) that (1.3) is satisfied for $i = 0$ and $i = 3$. Thus, we have proven that there exist at least two solutions to (1.2)–(1.3). Note that it is more convenient to vary the point u_0 rather than the quantities λ_1 and λ_2 when constructing an example of nonuniqueness for the piecewise-constant function $f(u)$.

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26 July 1965 Moscow